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An Isomorphism Theory for Bernoulli Free Z-Skew-Compact Group Actions

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I. INTRODUCTION

What we wish to consider here are measure-preserving actions of a group G, where G can be written as a skew product of Z with some compact metrizable group $\bar{G}$. Thus G can be written as $\{(n, \bar{g}) \mid n \in Z, \bar{g} \in \bar{G}\}$, where $(n, \bar{g}) \circ (n', \bar{g}') = (n + n', \phi^{n'}(\bar{g}) \circ \bar{g}')$, ϕ a continuous automorphism of $\bar{G}$. In this case we will write $G = Z \otimes^{\phi} \bar{G}$.

An "action" of G is a collection of measure preserving transformations $\{T_g\}_{g \in G}$, of a nonatomic Lebesgue space $(\Omega, \mathscr{F}_\mu)$, where $T_g \circ T_{g'} = T_{g \circ g'}$, and $T_g(\omega)$ as a map from $G \times \Omega \to \Omega$ is measurable.

With certain restrictions, what we wish to prove is that any two $Z \otimes^{\phi} \bar{G}$ actions whose Z-subgroup actions are Bernoulli shifts of the same entropy, are isomorphic. This argument will involve translating to this case all the machinery of Ornstein's isomorphism theory [3], [5]. We will assume the reader is familiar with this material, and when the translation is clear, will omit it, or refer the reader. D. Lind has already proven this result when ϕ is an ergodic automorphism of a torus (this is as yet unpublished but is discussed in [2]).

Any $G = Z \otimes^{\phi} \bar{G}$ action $\{T_g\}_{g \in G}$ on $(\Omega, \mathscr{F}, \mu)$ can be given a basic representation as follows. Let $\mathscr{A}(T_g)$ be the σ-algebra of ergodic components of the $\bar{G}$-action $\{T_{(0,\bar{g})}\}_{\bar{g} \in \bar{G}}$. This algebra is $T_{(1,0)} = T_1$ invariant. Let $\{\bar{T}, \bar{\Omega}, \bar{\mathscr{F}}, \bar{\mu}\}$ be a representation as a point mapping of the factor map $T_1/\mathscr{A}(T_g)$. The fiber over a point of this factor $\bar{\omega} \in \bar{\Omega}$ is an ergodic component for $\{T_{(0,\bar{g})}\}_{\bar{g} \in \bar{G}}$. Any such must be a coset space $\{H_{\bar{\omega}}\, g\} = F_{\bar{\omega}}$, where $H_{\bar{\omega}}$ is the isotropy subgroup of some point in the fiber. Now $\Omega = \bigoplus_{\bar{\omega} \in \bar{\Omega}} F_{\bar{\omega}}$, where the action of $\{T_g\}_{g \in G}$ is given by

$$T_{\bar{g}}(\bar{\omega}, H_{\bar{\omega}}\, \bar{g}') = (\bar{\omega}, H_{\bar{\omega}}\, \bar{g}' \circ \bar{g}) \qquad \text{for} \quad \bar{g} \in \bar{G}, \tag{1.1}$$

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and

$$T_1(\bar\omega, H_{\bar\omega}\bar g') = (\bar T(\bar\omega), H_{\bar T(\bar\omega)}\, g_{\bar\omega} \circ \phi^{-1}(\bar g')),$$

where $g_{\bar\omega}\colon \bar\Omega \to \bar G$ is a measurable map. This implies the compatability condition

$$g_{\bar\omega}^{-1}\phi(H_{\bar\omega})\, g_{\bar\omega} = H_{T(\bar\omega)}.$$

In this general form we cannot prove our isomorphism result, and, in fact, some clever examples due to J. Feldman (to appear separately) show the result to be false in this context. What we must assume is that

$$\bar\mu\{\bar\omega \mid H_{\bar\omega} \neq \{I\}\} = 0, \tag{1.2}$$

i.e., after deleting a set of measure zero, the fibers $F_\omega = \bar G$. We will say $\{T_g\}_{g\in G}$ is a free G-action if (1.2) is satisfied. In this case

$$\Omega = \bar\Omega \times \bar G, \qquad \mu = \bar\mu \times \nu$$

where ω is normalized Haar measure and

$$\begin{aligned} T_{\bar g}(\bar\omega, \bar g') &= (\bar\omega, \bar g' \circ \bar g), \\ T_1(\bar\omega, \bar g') &= (\bar T(\bar\omega), g_{\bar\omega} \circ \phi^{-1}(\bar g')), \end{aligned} \tag{1.3}$$

where $g_{\bar\omega}\colon \bar\Omega \to \bar G$ is a measurable map.

Thus a free G-action is determined by $(\bar T, \bar\Omega, \bar{\mathscr F}, \mu)$, and the map $g_{\bar\omega}\colon \bar\Omega \to \bar G$. We will abbreviate this representation by $(\bar T, g_{\bar\omega})$, and when we write T_g, $g \in G$, it will be as defined in (1.3).

In terms of this representation, two free G-actions $(\bar T, g_{\bar\omega})$ and $(\bar T', g'_{\bar\omega})$ are isomorphic iff there exists a measure preserving invertible map $\bar\psi\colon \bar\Omega \to \bar\Omega$ and an $\alpha_{\bar\omega}\colon \bar\Omega \to \bar G$ so that

$$\bar\psi \bar T \bar\psi^{-1} = \bar T', \tag{1.4}$$

and

$$g'_{\bar\psi(\bar\omega)} = \alpha_{\bar T(\bar\omega)} \circ g_{\bar\omega} \circ \phi^{-1}(\alpha_{\bar\omega}^{-1}).$$

If (1.4) is satisfied, the map $\psi(\bar\omega, \bar g) = (\bar\psi(\bar\omega), \alpha_{\bar\omega} \circ g)$ is an isomorphism, and if they are isomorphic, $\bar\psi$ is the restriction of the isomorphism taking $\mathscr{O}(T_g)$ to $\mathscr{O}(T'_g)$, and $\alpha_{\bar\omega}$ is the relabeling map on the fiber. Thus, isomorphism amounts to solving the functional equation (1.4).

Thus, in order to prove an isomorphism theorem we must construct $\bar\psi$ and $\alpha_{\bar\omega}$. Building $\bar\psi$ is precisely what the usual isomorphism theory does. What we will show is that simultanously we can build $\alpha_{\bar\omega}$.

We must first translate the basic structure of the isomorphism theorem into the context of this representation.

II. Lifting the Basic Structure from Z to $Z \otimes^{\phi} \bar{G}$

(a) *G-process*: A G-process, $G = Z \otimes^{\phi} \bar{G}$, will be a free G-action $(\bar{T}, g_{\bar{\omega}})$ along with a partition P in $\bar{\mathscr{F}} = \mathcal{O}\!\ell(T_g)$. It is not necessary that P generate $\bar{\mathscr{F}}$ under the action of $\bar{T}$. If it does, we will call the process "generating." We abbreviate a G-process by $(\bar{T}, P, g_{\bar{\omega}})$.

(b) *Entropy*: The entropies we are interested in are $h(\bar{T}) = h(T_1/\mathcal{O}\!\ell(T_g)$ and $h(T_1)$. We first prove a lemma showing how these two are related.

Lemma 1. *Let $(\bar{T}, g_{\bar{\omega}})$ be a free G-action, $G = Z \otimes^{\phi} \bar{G}$. We have then*

$$h(T_1) = h(\bar{T}) + h(\phi),$$

where $h(\phi)$ is the entropy of ϕ as a ν-preserving map on $\bar{G}$.

Proof. Write $T_1(\bar{\omega}, \bar{g}) = (\bar{T}(\bar{\omega}), g_{\bar{\omega}} \circ \phi^{-1}(\bar{g}))$. Define two new maps,

$$\bar{\Omega} \times \bar{G} \times \bar{G}$$

to itself;

$$\hat{T}(\bar{\omega}, \bar{g}_1, \bar{g}_2) = (\bar{T}(\bar{\omega}), g_{\bar{\omega}} \circ \phi^{-1}(\bar{g}_1), g_{\bar{\omega}} \circ \phi^{-1}(\bar{g}_2)),$$

and

$$\hat{T}'(\bar{\omega}, \bar{g}_1, \bar{g}_2) = (\bar{T}(\bar{\omega}), g_{\bar{\omega}} \circ \phi^{-1}(\bar{g}_1), \phi^{-1}(\bar{g}_2)).$$

Now $\hat{T}' = T_1 \times \phi^{-1}$, hence

$$h(\hat{T}') = h(T_1) + h(\phi). \tag{2.1}$$

To compute $h(\hat{T})$, for any partition Q of $\bar{G}$, let $Q_1 = \bar{\Omega} \times Q \times \bar{G}$ and $Q_2 = \bar{\Omega} \times \bar{G} \times Q_2$, and notice that for all N,

$$\bigvee_{i=0}^{N} \hat{T}^i(Q_1) \overset{\bar{\mathscr{F}} \times \bar{G} \times \bar{G}}{\perp} \bigvee_{i=0}^{N} \hat{T}^i(Q_2).$$

Hence

$$h(\hat{T}) = h(T_1/\mathcal{O}\!\ell T_g) + 2h(T_1 \mid \mathcal{O}\!\ell(T_g)),$$

and

$$h(\hat{T}) = h(T_1) + (h(T_1) - h(\bar{T})). \tag{2.2}$$

But $\hat{T}$ and $\hat{T}'$ are isomorphic by the map $\hat{\psi}$; $(\bar{\omega}, \bar{g}_1, \bar{g}_2) \to (\bar{\omega}, \bar{g}_1, \bar{g}_1 \circ \bar{g}_2)$. Hence $h(\hat{T}) = h(\hat{T}')$, and $h(T_1) = h(\bar{T}) + h(\phi)$. ∎

We need to gain some deeper understanding of

$$h(T_1 \mid \mathcal{Cl}(T_g)) = h(\phi).$$

What we want to do is give this a topological definition. This amounts to "relativizing" Propositions 7 and 9 of [1].

For a point $\bar{\omega} \in \bar{\Omega}$, and partition P of $\bar{\Omega} \times \bar{G}$, we can define

$$h_{\bar{\omega}}(T_1, P) = \lim_{n \to \infty} \frac{1}{n} h\left(\bigvee_{i=1}^{n} T_1^i(P)/\bar{\omega} \times \bar{G}\right), \tag{2.3}$$

and it is standard that

$$h(T, P \mid \mathcal{Cl}(T_g)) = \int_{\bar{\Omega}} h_x(T_1, P)\, d\bar{\mu}. \tag{2.4}$$

As in [1], we can give a different construction. For each $\bar{\omega}$, define $r^{\bar{\omega}}_{T_1}(\varepsilon, n)$ as the card of the smallest set $M \subset \bar{G}$ so that for any $\bar{g} \in \bar{G}$, there is a $\bar{g}' \in M$ with

$$\| \bar{g}(T_1^i(\bar{\omega}, \bar{g})), \bar{g}(T_1^i(\bar{\omega}, \bar{g}'))\| < \varepsilon, \tag{2.5}$$

where $\| \cdot , \cdot \|$ is the shift invariant metric on $\bar{G}$, and $\bar{g}((x, \bar{g}')) = \bar{g}'$ is the second coordinate of a point.

Now (2.5) is equal to

$$\begin{aligned}&\| g_{T_1^{i-1}(\omega)} \circ \phi(g_{T_1^{i-2}(\omega)}) \circ \cdots \circ \phi^{i-1}(g_{\bar{\omega}}) \circ \phi^i(\bar{g}),\\ &\qquad g_{T_1^{i-1}(\bar{\omega})} \circ \phi(g_{T_1^{i-2}(\bar{\omega})}) \circ \cdots \circ \phi^{i-1}(g_{\bar{\omega}}) \circ \phi^i(\bar{g}')\| = \|\phi^i(\bar{g}), \phi^i(\bar{g}')\|.\end{aligned}$$

Thus,

$$r^{\bar{\omega}}_{T_1}(\varepsilon, n) = r_\phi(\varepsilon, n). \tag{2.6}$$

Set

$$\begin{aligned} r^{\bar{\omega}}_{T_1}(\varepsilon) &= \limsup_{n \to \infty} \frac{1}{n} \log[r^{\bar{\omega}}_{T_1}(\varepsilon, n)] \\ &= \limsup_{n \to \infty} \frac{1}{2} \log[r_\phi(\varepsilon, n)] \\ &= r_\phi(\varepsilon). \end{aligned} \tag{2.7}$$

Now Proposition 9 of [1] says

$$\lim_{\varepsilon \to 0} r_\phi(\varepsilon) = h(\phi). \tag{2.8}$$

Thus we get

LEMMA 2. *For any free $Z \otimes^{\phi} \bar{G}$ action $(\bar{T}, \bar{g}_{\omega})$,*

$$h(T_1 \mid \mathcal{A}(T_y)) = \lim_{\varepsilon \to 0} \int_{\bar{\Omega}} r_{T_1}^{\bar{\omega}}(\varepsilon)\, d\bar{\mu},$$

where the rate of convergence is independent of $(\bar{T}, \bar{g}_{\omega})$. ∎

Thus we can modify Proposition 7 of [1] to get the following results.

LEMMA 3. *For any $\varepsilon > 0$, there is a $\delta(\varepsilon)$ so that if Q is any partition of $\bar{\Omega} \times \bar{G}$ so that $Q \cap (\bar{\omega} \times \bar{G})$ is made of sets of diameter less than $\delta(\varepsilon)$, and $(\bar{T}, \bar{g}_{\omega})$ is any free $Z \otimes^{\phi} \bar{G}$ action, then*

$$h(T_1, \bar{\Omega} \times Q \mid \mathcal{A}(T_g)) \geqslant h(\phi) - \varepsilon.$$

Proof. See the proof of Proposition 7 of [1].

LEMMA 4. *For any $\varepsilon > 0$, there is a $\hat{\delta}(\varepsilon)$ that for any $Z \otimes^{\phi} \bar{G}$ action $(\bar{T}, \bar{g}_{\omega})$, and any partition P of $\bar{\Omega} \times \bar{G}$, so that for some N, $\bigvee_{i=-N}^{N} T_1^i(P) \cap (\bar{\omega} \times \bar{G})$, for all but $\hat{\delta}(\varepsilon)$ of the $\bar{\omega} \in \bar{\Omega}$, is, after deleting $\hat{\delta}(\varepsilon)$ of $(\bar{\omega} \times \bar{G})$, made up of atoms of diameter less than $\hat{\delta}(\varepsilon)$, then*

$$h(T_1, P \mid \mathcal{A}(T_g)) \geqslant h(\phi) - \varepsilon.$$

Proof. Let $S \subset \bigvee_{i=-N}^{N} T^i(P)$ be a partition so that for all but a set $C_{\bar{\omega}} \subset \bar{\omega} \times \bar{G}$, $S \cap C_{\bar{\omega}}^c \cap (\bar{\omega} \times \bar{G})$ has atoms of diameter at most $\hat{\delta}$, and for all but $\hat{\delta}$ of the $C_{\bar{\omega}}$, $\nu(C_{\bar{\omega}}) < \hat{\delta}$. Partition $C = \bigcup_{\bar{\Omega}} C_{\bar{\omega}}$ into sets which, on each fiber, have diameter at most $\delta(\varepsilon/2)$. Call this partition $\tilde{C}$. No matter how small C is, $\tilde{C}$ can be chosen with a bounded number of elements. Thus, if $\hat{\delta}$ is small enough

$$h(T_1, \tilde{C} \mid \mathcal{A}(T_g)) < \frac{\varepsilon}{2}.$$

By Lemma 2,

$$h(T_1, P \vee \tilde{C} \mid \mathcal{A}(T_g)) > h(\phi) - \frac{\varepsilon}{2},$$

and hence

$$h(T_1, P \mid \mathcal{A}(T_g)) > h(\phi) - \varepsilon. \quad ∎$$

Thus, just as Ω split into a probabilistic part, $\bar{\Omega}$, and a topological group part, $\bar{G}$, as did an isomorphism ψ into $\bar{\psi}$ and α_ω, so also $h(T_1)$ splits into two pieces, one measure theoretic, the other topological.

(c) *Distributions*: Suppose $(\Omega, \mathscr{F}, \mu)$ and $(\Omega', \mathscr{F}', \mu')$ are two probability spaces, and $(M, \|\cdot,\cdot\|)$ is a compact metric space. Let $g: \Omega \to M$ and $g': \Omega' \to M$ be such that inverse images of open sets are measurable. We will say

$$\operatorname*{dist}_{\Omega}(g) = \operatorname*{dist}_{\Omega'}(g')$$

iff there is a probability space $(X, \mathscr{G}, \mathscr{N})$ and measure preserving maps Π_1: $X \to \Omega$, $\Pi_2: X \to \Omega'$ so that $\int_X \|g(\Pi_1(x)), g'(\Pi_2(x))\| = 0$. It is not too difficult to check that this is an equivalence relation.

If we define a finite partition P as a map to a finite set of names with the discrete metric, this reduces to the usual notion. If the measures given on Ω and Ω' are finite, but not normalized, replace them in this definition by their normalizations. Now define

$$|\operatorname*{dist}_{\Omega}(g), \operatorname*{dist}_{\Omega'}(g')| = \inf_{X, \Pi_1, \Pi_2} \int_X \|g(\Pi_1(x)), g'(\Pi_2(x))\|\, d\eta. \tag{2.9}$$

For two maps $g: \Omega \to M$, $g': \Omega \to M'$, define $g \vee g'$; $\Omega \to M \times M'$ by $(g \vee g')(\omega) = (g(\omega), g'(\omega))$, where $M \times M'$ has the sup metric.

If $\sum \lambda_i = 1$, $g_i: \Omega_i \to M$, we define $\sum \lambda_i \operatorname{dist}_{\Omega_i}(g_i)$ as the dist over $\bigcup \Omega_i$ of the map $\bar{g}$, $\bar{g}(\omega) = g_i(\omega)$ if $\omega \in \Omega_i$, where the measure μ on $\bigcup_i \Omega_i$ is $\sum \lambda_i \mu_i$. Finally, if $T: \Omega \to \Omega'$, $g: \Omega' \to M$, then define $T(g): \Omega \to M$ by $T(g(\omega)) = g(T^{-1}(\omega))$. (It is worthwhile to check that all of this is a lifting to the case of arbitrary g, the notion of a finite partition.) In our case, M will be a finite product of finite partition labels and copies of $\bar{G}$, and the map $g(\omega)$ will be the partition name of ω and the trajectory of ω through $\bar{G}$ under the action of T_1.

The ergodic Theorem and the strong Rochlin Theorem are results about distributions which are essential to the Isomorphism Theorem for Z actions. Here are the versions we want of them. The proofs are minor modifications of the standard ones, hence we omit proofs.

LEMMA 5. *Let T be an ergodic measure preserving transformation of $(\Omega, \mathscr{F}, \mu)$, and $g: \Omega \to M$, $(M, \|\cdot,\cdot\|)$ a compact metric space. For any $\varepsilon > 0$, then, there is an N and a set A, $\mu(A) > 1 - \varepsilon$, so that if $\omega \in A$, and $n > N$, then*

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \operatorname*{dist}_{T^i(\omega)}(g), \operatorname*{dist}_{\Omega}(g) \right| < \varepsilon. \quad \blacksquare$$

LEMMA 6. *Let T be an ergodic measure preserving transformation of $(\Omega, \mathscr{F}, \mu)$, and $g: \Omega \to M$, $(M, \| \cdot, \cdot \|)$ a compact metric space. Given any $N > 0$ and $\varepsilon > 0$, there is a set $F \in \mathscr{F}$, so that F, $T(T)$,..., $T^{N-1}(F)$ are disjoint and span all but ε of Ω, and*

$$\left| \operatorname*{dist}_F \left(\bigvee_{i=0}^{N-1} T^{-i}(g) \right), \operatorname*{dist}_\Omega \left(\bigvee_{i=0}^{N-1} T^{-i}(g) \right) \right| < \varepsilon. \quad \blacksquare$$

We will also need a simple copying Lemma concerning distributions, that says closeness in distribution can be nearly achieved by a map $\Omega \to \Omega'$.

LEMMA 7. *Let $(\Omega, \mathscr{F}, \mu)$ and $(\Omega', \mathscr{F}', \mu')$ be two probability spaces, Ω a nonatomic Lebesgue space. Let $g: \Omega \to M$, $g': \Omega' \to M$, $(M, \| \cdot, \cdot \|)$ a compact metric space. Given any $\varepsilon > 0$, there is a measure preserving map $\Pi: \Omega \to \Omega'$ so that*

$$\int_\Omega \| g'(\Pi(\omega)), g(\omega) \| \, d\mu < |\operatorname*{dist}_\Omega (g), \operatorname*{dist}_{\Omega'} (g')| + \varepsilon.$$

Proof. Let $(x, \mathscr{G}, \mathscr{N})$, Π_1, Π_2 be a joining which achieves, to within $\varepsilon/2$, the dist distance between g and g'. Let S be a partition of M into G_δ sets of diameter at most $\varepsilon/2$. For any $S_i \in S$,

$$\mu(g^{-1}(S_i)) = \mathscr{N}(\Pi_1^{-1}(g^{-1}(S_i))).$$

Let $A_i = g^{-1}(S_i)$ and $B_i = \Pi_1^{-1}(g^{-1}(S_i))$. As Ω is nonatomic, we can define a measure preserving map $\bar{\Pi}: A_i \to B_i$ for all i. Now

$$\int_\Omega \| g(\omega), g(\Pi_1 \circ \bar{\Pi}(\omega)) \| \, d\mu < \frac{\varepsilon}{2}.$$

Let $\Pi = \Pi_2 \circ \bar{\Pi}$ and the result follows. $\blacksquare$

We will say two G-processes $(\bar{T}, P, g_{\bar{\omega}})$ and $(\bar{T}', P', g'_{\bar{\omega}'})$ are equal as processes when $g_{\bar{\omega}} \subset \bigvee_{i=-\infty}^{\infty} \bar{T}^i(P)$ and $g'_{\bar{\omega}'} \subset \bigvee_{i=-\infty}^{\infty} \bar{T}'^i(P')$, and for all n

$$\operatorname*{dist}_{\bar{\Omega}} \bigvee_{i=0}^{n-1} \bar{T}^i(P \vee g_{\bar{\omega}}) = \operatorname*{dist}_{\bar{\Omega}'} \bigvee_{i=0}^{n-1} \bar{T}'^i(P' \vee g'_{\bar{\omega}'}).$$

If this is the case, and the processes are generated, it is clear that $\{T_g\}_{g \in G}$ and $\{T'_g\}_{g \in G}$ are isomorphic.

(d) *G-finitely determined*: Our next step is to define a $\bar{d}$ metric adapted to $Z \otimes^\phi \bar{G}$-processes, and with it, a notion of finitely determined. If we have two G-processes $(\bar{T}, P, g_{\bar{\omega}})$ and $(\bar{T}', P', g'_{\bar{\omega}'})$ that are close in this metric, it will mean that T_1, P-names and T'_1, P'-names of points can be matched in the usual $\bar{d}$, but further, under this match, trajectories of points through $\bar{g}$ are close, on the average, in the group metric. More precisely, consider measure preserving onto maps $\Pi_1: X \to \bar{\Omega} \times \bar{G}$, $\Pi_2: X \to \bar{\Omega}' \times \bar{G}$, $(X, \mathscr{S}, \mathscr{N})$ some Lebesgue space. Now

$$\begin{aligned} &\bar{d}_n^G((\bar{T}, P, g_{\bar{\omega}}); (\bar{T}', P', g'_{\bar{\omega}'})) \\ &= \inf_{X, \Pi_1, \Pi_2} \left(\frac{1}{n} \sum_{i=0}^{n-1} |\Pi_1^{-1}(T_1^i(P)), \Pi_2^{-1}(T_1'^i(P'))|_{\mathscr{N}} \right. \\ &\quad \left. + \frac{1}{n} \sum_{i=0}^{n-1} \int_X \| \bar{g}(T_1^i(\Pi_1(x))), \bar{g}(T_1^i(\Pi_2(x)))\| \, d\mathscr{N}, \right. \end{aligned} \tag{2.10}$$

($\bar{g}(\ )$ is always the map to the second coordinate).

Normally $\bar{d}^G$ would now be defined as $\limsup_n \bar{d}_n^G$, but we will instead give a much stronger definition, lifting the strongest notion of $\bar{d}$. An "ergodic joining" for (T, g_ω) will be a Lebesgue space $(X, \mathscr{S}, \mathscr{N})$ with a free G-action $\{\hat{T}_g\}_{g \in \underline{G}}$ on it, with measure preserving maps $\Pi_1: X \to \bar{\Omega} \times \bar{G}$ and $\Pi_2: X \to \bar{\Omega}' \times \bar{G}$ with $\Pi_1 \hat{T}_g = T_g \Pi_2$ and $\Pi_2 \hat{T}_g = T'_g \Pi_2$ for all $g \in G$, and where $(\hat{T}_1, \Pi_1^{-1}(\mathscr{O}(T_g)) \vee \Pi_2^{-1}(\mathscr{O}(T'_g)))$ is ergodic. Now

$$\begin{aligned} &\bar{d}^G((\bar{T}, P, g_{\bar{\omega}}); (\bar{T}', P', g'_{\bar{\omega}'})) \\ &= \inf_{\text{ergodic joinings}} \left(|\Pi_1^{-1}(P), \Pi_2^{-1}(P')| + \int_X \| \bar{g}(\Pi_1(x)), \bar{g}(\Pi_2(x))\| \, d\mathscr{N} \right). \end{aligned} \tag{2.11}$$

If $(\bar{T}, P, g_{\bar{\omega}})$ and $(\bar{T}', P', g'_{\bar{\omega}'})$ are generated G-processes and $\bar{d}^G((\bar{T}, P, g_{\bar{\omega}}); (\bar{T}', P', g'_{\bar{\omega}'})) = 0$, it is clear that for all n,

$$\operatorname*{dist}_{\bar{\Omega}} \left(\bigvee_{i=0}^{n-1} \bar{T}^i(P \vee g_{\bar{\omega}}) \right) = \operatorname{dist} \left(\bigvee_{i=0}^{n-1} \bar{T}'^i(P' \vee g'_{\bar{\omega}'}) \right),$$

hence $(\bar{T}, P) = (\bar{T}', P')$ as processes, but moreover, $g_{\bar{\omega}}$ and $g_{\bar{\omega}'}$ are generated by each in the same way. Hence the G-actions are isomorphic.

We will say a generated G-process $(\bar{T}, P, g_{\bar{\omega}})$ is "G-finitely determined" if for any $\varepsilon > 0$, there is a $\delta > 0$ and an N so that for any ergodic G-process $(\bar{T}', P', g'_{\bar{\omega}'})$, $\operatorname{card}(P) = \operatorname{card}(P')$ and

(i) $|h(\bar{T}, P) - h(\bar{T}', P')| < \delta$ and $|h(\bar{T}', P') - h(\bar{T}')| < \delta$

(ii) $\left| \operatorname*{dist}_{\bar{\Omega} \times \bar{G}} \bigvee_{i=0}^{N-1} T_1^{-i}(P \vee \bar{g}), \operatorname*{dist}_{\bar{\Omega}' \times \bar{G}} \bigvee_{i=0}^{N-1} T_1^{1-i}(P' \vee \bar{g}) \right| < \delta,$

then

(iii) $\bar{d}^G((\bar{T}, P, g_{\bar{\omega}}); (\bar{T}', P', g'_{\bar{\omega}'}))$.

What we will show now is that if T_1 is finitely determined in the usual sense [3], then $(\bar{T}, P, g_{\bar{\omega}})$ is G-finitely determined.

THEOREM 1. *Let* $(\bar{T}, P, g_{\bar{\omega}})$ *be a generated G-process,* $G = Z \otimes^{\phi} \bar{G}$, *where* T_1 *is isomorphic to a Bernoulli shift of finite entropy.* $(\bar{T}, P, g_{\bar{\omega}})$ *is, then, G-finitely determined.*

Proof. Let $(\bar{T}, P, g_{\bar{\omega}})$ be as given. To verify G-finitely determined, we are given $\varepsilon > 0$ and must find δ and N. We know T_1, on any partition, is finitely determined. Pick, as a partition $(\bar{P} \vee \bar{Q} \vee H)$, where $\bar{P} = P \times \bar{G}$, $\bar{Q} = \bar{\Omega} \times \bar{Q}$, where Q is a partition of $\bar{G}$ into sets of diameter less than $\varepsilon/10$, and H is a generator for T_1 on Ω. This is now fixed.

There is, then, a δ, and N, so that if $(T_1, \bar{P}' \vee \bar{Q}' \vee H')$ is such that

(i)′ $|h(T_1', \bar{P}' \vee \bar{Q}' \vee H') - h(T_1, \bar{P} \vee \bar{Q} \vee H)| < \delta_1$,

(ii)′ $$\left| \operatorname*{dist}_{\bar{\Omega}' \times \bar{G}} \left(\bigvee_{i=1}^{N} T_1'^{-i}(\bar{P}' \vee \bar{Q}' \vee H') \right), \operatorname*{dist}_{\bar{\Omega} \times \bar{G}} \left(\bigvee_{i=1}^{N} T_1^{-i}(\bar{P} \vee \bar{Q} \vee H) \right) \right| < \delta_1,$$

then

(iii)′ $\bar{d}(T_1', \bar{P}' \vee \bar{Q}' \vee H'; T_1, \bar{P} \vee \bar{Q} \vee H) < \varepsilon/10$.

Now pick N_2 so that

$$\bigvee_{i=-N_2}^{N_2} T_1^i(H), \qquad \left(\frac{\hat{\delta}(\delta_1/10)}{10} \right)^2 \tag{2.12}$$

generates a partition $\bar{\Omega} \times S$, where S is made of sets of diameter less than $(\hat{\delta}(\delta_1/10))/10$ ($\hat{\delta}(\ )$ is defined in Lemma 4), and $N_1/N_2 < (\hat{\delta}(\delta_1/10))/10$. Now set

$$N = \frac{10 N_2}{\hat{\delta}(\delta_1/10)}. \tag{2.13}$$

Now pick δ so small that for any $\bar{g} \in \bar{G}$, $\| \bar{g}, I \| < \delta$,

$$| T_{\bar{g}}(\bar{P} \vee \bar{Q} \vee H), \bar{P} \vee \bar{Q} \vee H | < \delta_1/10, \tag{2.14}$$

and

$$\delta < \left(\frac{\hat{\delta}(\delta_1/10)}{10} \right)^2.$$

Now take any ergodic $(\bar{T}', P', g'_{\bar{\omega}'})$, $\bar{T}'$ aperiodic with

$$\text{(i)}\quad |h(\bar{T}, P) - h(\bar{T}', P')| < \delta, \qquad |h(\bar{T}', P') - h(\bar{T}')| < \delta,$$

$$\text{(ii)}\quad \left| \operatorname*{dist}_{\bar{\Omega}\times\bar{G}} \bigvee_{i=0}^{N-1} T_1^{-i}(P \vee \bar{g}), \operatorname*{dist}_{\bar{\Omega}'\times\bar{G}} \bigvee_{i=0}^{N-1} T_1'^{-i}(P' \vee \bar{g}') \right| < \delta.$$

Set $\bar{P}' = P' \times \bar{G}$, $\bar{Q}' = \bar{\Omega}' \times Q$. We need to define an H'. To do this, let X, Π_1, Π_2 be the joining that gives (ii). Use Lemma 6 with N, $\varepsilon = \delta$ and $g = \bigvee_{i=0}^{N-1} T_1^{1-i}(\bar{P} \vee \bar{g})$ to select F. Using Lemma 7, we can build a map $\Pi: \bar{\Omega} \times \bar{G} \to F$ so that

$$\frac{1}{\mu(F)} \left| \bigvee_{i=0}^{N-1} T_1'^{-i}(P \vee \bar{g})/F, \bigvee_{i=0}^{N-1} \Pi(T_1^i(P \vee \bar{g})) \right|_{\mu/F} + \frac{1}{\mu(F)} \int_F \sup_{0 \leqslant i \leqslant N} (\| \bar{g} T_1^{-i}(\Pi(\omega)), \bar{g}(T_1^{-i}(\omega)) \|) \, d\mu < 2\delta. \tag{2.15}$$

Now define

$$H' = \bigcup_{i=0}^{N-1} T'^i(\Pi(T^{-i}(H)))/T^i(F).$$

From (2.12), (2.14) and (2.15),

$$\left| \operatorname*{dist}_{\bar{\Omega}'\times\bar{G}} \bigvee_{i=0}^{N_1-1} T_1^{-i}(\bar{P}' \vee \bar{Q}' \vee H'), \operatorname*{dist}_{\bar{\Omega}\times\bar{G}} \bigvee_{i=0}^{N_1-1} T_1^{-i}(\bar{P} \vee \bar{Q} \vee H) \right| < 4\delta + \frac{\delta_1}{5} < \delta_1.$$

This is (i)$'$.

To get (ii)$'$, notice, that by (2.15), and the size of N_2, on all but $(2\hat{\delta}(\delta_1/10))/10$ of the $\bar{\omega}' \in \bar{\Omega}'$, $\bigvee_{i=-N_2}^{N_2} T_1'^i(H')/(\bar{\omega} \times \bar{G})$ generates a partition S' (the analogue of S) so that on all but $(\hat{\delta}(\delta_1/10))/10$ of $\bar{G}$, the atom S' have diameter at most $(3\hat{\delta}(\delta_1/10))/5$. Thus by Lemma 4, and (ii)

$$|h(T_1', \bar{P}' \vee \bar{Q}' \vee H') - h(T_1, \bar{P} \vee \bar{Q} \vee H)| < 2\delta + \frac{\delta_1}{10} < \delta_1.$$

This is (ii)$'$. Hence we get

$$\bar{d}(T_1, \bar{P} \vee \bar{Q}; T_1', \bar{P}' \vee \bar{Q}') < \varepsilon/10, \tag{2.16}$$

(we leave out H and H' as they are no longer needed).

This means there is an ergodic $T_1 \times T_1'$ invariant measure $\hat{\mu}$ on $\Omega \times \Omega'$, where $\hat{\mu}$ projects to μ and μ' on the coordinate algebras and

$$|\bar{P} \vee \bar{Q} \times \Omega', \Omega \times \bar{P}' \vee \bar{Q}'|_{\hat{\mu}} < \frac{\varepsilon}{10},$$

(see Appendix C to [4] for this version of $\bar{d}$). Now $\hat{\mu}$ is not $T_g \times T'_g$ invariant for all $g \in G$, but the two marginals are. So define $\bar{\mu}$ by

$$\bar{\mu}(A) = \int_{\bar{g} \in \bar{G}} \left(\int_{\Omega \times \Omega'} (T_{\bar{g}} \times T'_{\bar{g}}(\chi_A))\, d\hat{\mu} \right) dv. \tag{2.17}$$

Now $\bar{\mu}$ is invariant under all $T_g \times T_g$, $g \in G$, the marginals on coordinates are still μ and μ', and as $\bar{\mu} = \hat{\mu}$ on $\mathcal{O}\!\ell(T_g) \times \mathcal{O}\!\ell(T'_g)$, $\bar{\mu}$ on $(T_1 \times T_1, (\bar{P} \times \Omega') \vee (\Omega \times P'))$ is ergodic. Thus $(\Omega \times \Omega', \mathscr{F} \times \mathscr{F}', \bar{\mu})$ with G-action $\{T_g \times T'_g\}_{g \in G}$ is an ergodic joining. We want to check that it makes $\bar{d}^G$ small. First $|\bar{P} \times \Omega', \Omega \times \bar{P}'|_{\bar{\mu}} = |\bar{P} \times \Omega', \Omega \times \bar{P}'|_{\hat{\mu}} < \varepsilon/10$. For the second term of (2.11),

$$\int_{x \in \Omega \times \Omega'} \int_{\hat{g} \in \bar{G}} \| \bar{g}(\hat{g} \circ \Pi_1(x)), \bar{g}(\hat{g} \circ \Pi_2(x))\| \, dv\, d\hat{\mu}$$

$$= \int_{x \in \Omega \times \Omega'} \| \bar{g}(\Pi_1(x)), \bar{g}(\Pi_2(x))\| \, d\hat{\mu}$$

as $\| \cdot, \cdot \|$ is shift invariant. As $|\bar{Q} \times \Omega', \Omega \times \bar{Q}'|_{\hat{\mu}} < \varepsilon/10$, for all but $\varepsilon/10$ of the $x \in \Omega \times \Omega'$, $\bar{g}(\Pi_1(x))$ and $\bar{g}(\Pi_2(x))$ are in the same set in Q. Hence

$$\| \bar{g}(\Pi_1(x)), \bar{g}(\Pi_2(x))\| < \varepsilon/10$$

and

$$\int_{\Omega \times \Omega'} \| \bar{g}(\Pi_1(x)), \bar{g}(\Pi_2(x))\| \, d\hat{\mu} < \varepsilon/5$$

and so

$$\bar{d}^G((\bar{T}, P, g_{\bar{\omega}}), (\bar{T}', P', g'_{\bar{\omega}'})) < 3\varepsilon/10. \quad \blacksquare$$

III. The Isomorphism Theorem

We are now ready to show that any two G-finitely determined free G-actions with the same finite entropy are isomorphic. More precisely, we want to show that given any two G-finitely determined generated G-process $(\bar{T}, P, g_{\bar{\omega}})$ and $(\bar{T}', P', g'_{\bar{\omega}'})$ with $h(T_1, P) = h(T'_1, P')$, there is a $\bar{P}' \subset \mathcal{O}\!\ell(T_g)$ and an $\alpha_{\bar{\omega}}: \bar{\Omega} \to G$ so that

$$(\bar{T}, \bar{P}', \alpha_{T(\bar{\omega})} \circ g_{\bar{\omega}} \circ \phi^{-1}(\alpha_{\bar{\omega}}^{-1})) \equiv (\bar{T}', P', g'_{\bar{\omega}'})$$

as processes, where $\bigvee_{i=-\infty}^{\infty} \bar{T}^i(\bar{P}') = \mathcal{A}(T_g)$. We will use the basic Isomorphism Theorem induction to construct $\bar{P}'$ and $\alpha_{\bar{\omega}}$. At the core of such arguments is always an imbedding Lemma. Arthur Rothstein [4] has noted that in the Z-action theory only one such is needed. We lift his idea and prove the following version of it.

LEMMA 8. *Let $(\bar{T}, P, g_{\bar{\omega}})$ be an ergodic generated G-process, and $(\bar{T}', P', g'_{\bar{\omega}'})$ an ergodic G-process, with*

$$\infty > h(T_1', P') = h(T_1/\mathcal{A}(T_g)) = h(T_1'/\mathcal{A}(T_g')) = h.$$

Let Π_1, Π_2 on $(X, \mathcal{G}, \mathcal{N})$ with G-action $\{T_g\}_{g\in G}$ be an ergodic joining for these two. For any $\delta > 0$ and N, there is a $\bar{P} \subset \mathcal{A}(T_g)$ and an $\bar{\alpha}: \bar{\Omega} \to \bar{G}$ so that

(i) $|h(T_1, \bar{P}') - h(T_1', P')| < \delta$,

(ii) $$\left| \operatorname*{dist}_X \left(\bigvee_{i=0}^{N-1} \hat{T}_1^{-i}(\Pi_1^{-1}(P \vee \bar{g}) \vee \Pi_2^{-1}(P' \vee \bar{g})) \right), \operatorname*{dist}_{\bar{\Omega}\times\bar{G}} \left(\bigvee_{i=0}^{N-1} T_1^{-i}((P \vee \bar{g}) \vee (\bar{P}' \vee \bar{\alpha} \circ \bar{g})) \right) \right| < \delta.$$

Proof. This is basically a Marriage Lemma argument. Choose a δ_1 whose precise size we set later. Let $N_1 = N/2\delta_1$ and now choose N_2, as the height of a Rochlin tower in $\mathcal{A}(T_g)$, F, $T_1(F) \cdots T_1^{N_2-1}(F)$ so that by Lemma 6,

$$\operatorname*{dist}_F \left(\bigvee_{i=0}^{N_2-1} T_1^{-i}(P) \right) = \operatorname*{dist}_{\bar{\Omega}\times\bar{G}} \left(\bigvee_{i=0}^{N_2-1} T_1^{-i}(P) \right), \tag{3.1}$$

and for any atom $E \subset \bigvee_{i=0}^{N_2-1} T_1^{-i}(P)$,

$$\left| \operatorname*{dist}_{E\cap F} \left(\bigvee_{i=0}^{N_2-1} T_1^{-i}(\bar{g}) \right), \operatorname*{dist}_E \left(\bigvee_{i=0}^{N_2-1} T_1^{-i}(\bar{g}) \right) \right| < \delta_1.$$

Keep in mind that in $\bar{\Omega} \times \bar{G}$, F is of the form $\bar{F} \times \bar{G}$.

Require

$$\text{all but } \delta_1 \text{ of the atoms } E' \subset \bigvee_{i=0}^{N_2-1} T_1^{-i}(P') \text{ and } E \subset \bigvee_{i=0}^{N_2-1} T_1^{-i}(P) \tag{3.2}$$

to have sizes within $2^{-N_2(h\pm\delta_1)}$, by using the Shannon McMillan Theorem,

$$\text{for all but } \delta_1 \text{ of the atoms } \hat{E} \subset \bigvee_{i=0}^{N_2-1} \hat{T}_1^{-1}(\Pi_1^{-1}(P) \vee \Pi_2^{-1}(P)), \tag{3.3}$$

$$\left| \frac{1}{N_2 - N} \sum_{j=0}^{N_2-N} \operatorname{dist}_{\hat{E}} \left(\sum_{i=0}^{N-1} \hat{T}_1^{-(j+i)}(\Pi_1^{-1}(P \vee \bar{g}) \vee \Pi_2^{-1}(P' \vee \bar{g})) \right), \right.$$
$$\left. \operatorname{dist}_{X} \left(\bigvee_{i=0}^{N-1} \hat{T}_1^{-i}(\Pi_1^{-1}(P \vee \bar{g}) \vee \Pi_2^{-1}(P' \vee \bar{g})) \right) \right| < \delta_1$$

by Lemma 5, and lastly

$$\text{for all but } \delta_1 \text{ of the atoms } E \subset \bigvee_{i=0}^{N_2-1} T_1^{-i}(P), \tag{3.4}$$

for $i = 0 \cdots N_1 - 1$ and $j = 0 \cdots N_2 - 1$, there are constants $g(i, j, E) \in \bar{G}$ so that

$$\frac{1}{N_2} \sum_{j=0}^{N_2-1} \left(\sum_{i=0}^{N_1-1} \int_{T_1^j(E)} \| \bar{g}(T_1^i(\omega)) \circ \phi^i(\bar{g}(\omega)), g(i, j, E) \| \right) < \delta_1 \mu(E).$$

As $\bar{g}(T_1^i(\omega)) \circ \phi^i(\bar{g}(\omega)) = g_{T^{i-1}(\bar{\omega})} \circ \phi^{-1}(g_{T^{i-2}(\bar{\omega})}) \circ \cdots \circ \phi^{-(i-1)}_{(g_{\bar{\omega}})}$ for any $\omega = (\bar{\omega}, g)$, this can be done if N_2 is large enough, as $g_{\bar{\omega}}$ is $\bigvee_{i=-\infty}^{\infty} T_1^i(P)$ measurable.

Now, if δ_1 is small enough, using the standard Marriage Lemma technique (see [3] or [5]) we can assign to each atom $E \subset \bigvee_{i=0}^{N_2} T_1^{-i}(P)$, the T_1', P', N_2-name of some atom $E'(E) \subset \bigvee_{i=0}^{N_2} T_1'^i(P')$, where $\hat{E}(E) = \Pi_1^{-1}(E) \cap \Pi_2^{-1}(E'(E)) \neq \varnothing$, and for all but $\delta/100$ of the E, $\hat{E}(E)$ satisfies (3.3), and from (3.4)

$$\frac{1}{N_2} \sum_{j=0}^{N_2-1} \left(\sum_{i=0}^{N_1-1} \int_{\hat{T}_1^j(\hat{E}(E))} \| \bar{g}(T_1^i(\Pi_1(x)) \circ \phi^i(\bar{g}(\Pi_1(x))), g(i, j, E) \| \right)$$
$$< \sqrt{\delta_1}\, (\hat{E}(E)). \tag{3.5}$$

Furthermore, by the usual argument, by assigning $E \cap F$ the T_1, $\bar{P}'$, N_2-name that is the T', P', N_2-name of $E'(E)$, we can get

(i) $|h(T, \bar{P}') - h| < \delta$.

We need to construct $\bar{\alpha}$ to get (ii). Now (3.3) already gives us half of (ii). Let E and $\hat{E}(E)$ satisfy (3.3) and (3.5). By (3.5) we can pick an L, $0 \leqslant L < N$ so that for a set $J(E)$ of all but $\sqrt[4]{2\delta_1}$ of the values $L, L + N_1 \cdots L + ([N_2/N_1] - 1) N_1$, if $j \in J(E)$, then

$$\sum_{i=0}^{N_1-1} \int_{\hat{T}_1^j(\hat{E}(E))} \| \bar{g}(T_1^i(\Pi_1(x))) \circ \phi^i(\bar{g}(\Pi_2(x))), g(i, j, E) \| < \sqrt[4]{2\delta_1}\, \mathscr{N}(\hat{E}(E)), \tag{3.6}$$

and by (3.1) and (3.4),

$$\sum_{i=0}^{N_1-1} \int_{T_1^i(E\cap F)} \| \bar{g}(T_1^i(\omega)) \circ \phi^i(\bar{g}(\omega)), g(i,j,E)\| < \sqrt[4]{2\delta_1}\, \mu(E \cap F).$$

Noticing that $\bar{g}(T_1^i(\Pi_1(x))) \circ \phi^i(\bar{g}(\Pi_2(x)))$ is $\hat{T}_{\bar{g}}$ invariant, $\bar{g} \in \bar{G}$, and $\bar{g}(T_1^i(\omega)) \circ \phi^i(\bar{g}(\omega))$ is $T_{\bar{g}}$ invariant, we get

$$\left| \operatorname*{dist}_{\hat{E}(E)} \left(\bigvee_{i=0}^{N_1-1} \hat{T}_1^{-(j+i)}(\Pi_1^{-1}(\bar{g})) \right), \operatorname*{dist}_{\bar{G}} \left(\bigvee_{i=0}^{N_1-1} g(i,j,E) \circ \phi^{-i} \right) \right| < \sqrt[4]{2\delta_1} \tag{3.7}$$

and

$$\left| \operatorname*{dist}_{E\cap F} \left(\bigvee_{i=0}^{N_1-1} T_1^{-(j+i)}(\bar{g}) \right), \operatorname*{dist}_{\bar{G}} \left(\bigvee_{i=0}^{N_1-1} g(i,j,E) \circ \phi^{-i} \right) \right| < \sqrt[4]{2\delta_1}. \tag{3.8}$$

Now take any $\bar{\alpha}$, $\mathcal{O}(T_g)$ measurable, on $\bigcup_{i=0}^{N_1} T^{j+i}(E \cap G)$ so that

$$\operatorname*{dist}_{\hat{E}(E)} \left(\bigvee_{i=0}^{N_1-1} \hat{T}_1^{-(j+i)}(\Pi_2^{-1}(\bar{g}) \circ (\Pi_1^{-1}(\bar{g}))^{-1}) \right) = \operatorname*{dist}_{E\cap F} \left(\bigvee_{i=0}^{N_1-1} T_1^{-(j+i)}(\bar{\alpha}) \right).$$

We can do this as $\Pi_2^{-1}(\bar{g}) \circ (\Pi_1^{-1}(\bar{g}))^{-1}$ is $\mathcal{O}(\hat{T}_g)$ measurable, and $E \cap F$ is nonatomic.

It now follows from (3.7) and (3.8) that

$$\left| \operatorname*{dist}_{\hat{E}(E)} \left(\bigvee_{i=0}^{N_1-1} \hat{T}_1^{-(j+i)}(\Pi_1^{-1}(\bar{g}) \vee \Pi_2^{-1}(\bar{g})) \right), \operatorname*{dist}_{E\cap F} \left(\bigvee_{i=0}^{N_1-1} T_1^{-(j+i)}(\bar{g} \vee \bar{\alpha} \circ \bar{g}) \right) \right|$$
$$< 2\sqrt[4]{2\delta_1},$$

and hence

$$\left| \frac{1}{N_1 - N + 1} \sum_{k=0}^{N_1-N} \left(\operatorname*{dist}_{\hat{E}(E)} \bigvee_{i=0}^{N} \hat{T}_1^{-(j+k+i)}(\Pi_1^{-1}(P \vee \bar{g}) \vee \Pi_2^{-1}(P' \vee \bar{g})) \right), \right.$$
$$\left. \frac{1}{N_1 - N + 1} \sum_{k=0}^{N_1-N} \left(\operatorname*{dist}_{E\cap F} \bigvee_{i=0}^{N} T_1^{-(j+k+i)}((P \vee \bar{g}) \vee (\bar{P}' \vee \alpha \circ \bar{g})) \right) \right|$$
$$< 2\sqrt[4]{2\delta_1}. \tag{3.9}$$

Do this for all $j \in J(E)$ and we conclude

$$\left| \frac{1}{N_2 - N + 1} \sum_{k=0}^{N_2-N} \left(\operatorname{dist}_{\hat{E}(E)} \bigvee_{i=0}^{N} \hat{T}_1^{-(k+i)}(\Pi_1^{-1}(P \vee \bar{g}) \vee \Pi_2^{-1}(P' \vee \bar{g})) \right), \right.$$
$$\left. \frac{1}{N_2 - N + 1} \sum_{k=0}^{N_2-N} \left(\operatorname{dist}_{E \cap F} \bigvee_{i=0}^{N} T_1^{-(k+i)}((P \vee \bar{g}) \vee (\bar{P}' \vee \alpha \circ \bar{g})) \right) \right|$$
$$< 3 \sqrt[4]{2\delta_1} + \delta_1. \tag{3.10}$$

Do this for all but the $\delta/100$ of the E for which $\hat{E}(E)$ is bad. Do these remaining arbitrarily, and (ii) follows from (3.3) and (3.10). ∎

We can now prove Sinai's Theorem for free G-actions.

THEOREM 2. *If $(\bar{T}, P, g_{\bar{\omega}})$ is a G-finitely determined generated G-process, and $(\bar{T}', P', g'_{\bar{\omega}'})$ is any ergodic generated G-process with $h(T_1') = h(T_1) < \infty$, then there is a $\bar{P}$ and $\alpha_{\bar{\omega}'}$, $\mathcal{O}(T_g')$ measurable, with*

$$(T', \bar{P}, \alpha_{\bar{T}'(\bar{\omega}')} \circ g'_{\bar{\omega}'} \circ \phi^{-1}(\alpha_{\bar{\omega}'}^{-1})) \equiv (\bar{T}, P, g_{\bar{\omega}})$$

as processes.

Proof. Start with any ergodic joining of $(\bar{T}, P, g_{\bar{\omega}})$ and $(\bar{T}', P', g'_{\bar{\omega}'})$ and a sequence $\varepsilon_i > 0$, $\sum \varepsilon_i < \infty$. Use Lemma 8, with an appropriate δ_1 and N_1, and the fact that $(\bar{T}, P, g_{\bar{\omega}})$ is G-finitely determined, to build $\bar{P}_1$, $\alpha_1 \subset \mathcal{O}(T_g')$ with

$$\bar{d}^G((\bar{T}, P, g_{\bar{\omega}}); (\bar{T}', \bar{P}_1, \alpha_{1\bar{T}'(\bar{\omega}')} \circ g'_{\bar{\omega}'} \circ \phi^{-1}(\alpha_{1\bar{\omega}'}^{-1}))) < \varepsilon_1.$$

Modify $\bar{P}_1$, using (i) of Lemma 8, with δ small enough, by less than ε_1, to $\hat{P}_1$, so that $\bigvee_{i=-\infty}^{\infty} T_1'^{i}(\hat{P}_1) = \mathcal{O}(T_g')$ and now

$$\bar{d}^G((\bar{T}, P, g_{\bar{\omega}}); (\bar{T}', \hat{P}_1, \alpha_{1\bar{T}'(\bar{\omega}')} \circ g'_{\bar{\omega}'} \circ \phi^{-1}(\alpha_{1\bar{\omega}'}^{-1}))) < 2\varepsilon_1.$$

Repeat this program using this joining of

$$(\bar{T}, P, g_{\bar{\omega}}) \qquad \text{and} \qquad (\bar{T}', \hat{P}_1, \alpha_{1\bar{T}'(\bar{\omega}')} \circ g'_{\bar{\omega}'} \circ \phi^{-1}(\alpha_{1\bar{\omega}'}^{-1})),$$

to build $\hat{P}_2$ and α_2 with

$$\bar{d}^G((\bar{T}, P, g_{\bar{\omega}}); (\bar{T}', \hat{P}_2, \alpha_{2\bar{T}'(\bar{\omega}')} \circ g'_{\bar{\omega}'} \circ \phi^{-1}(\alpha_{2\bar{\omega}'}^{-1}))) < 2\varepsilon_2,$$

and

$$|\hat{P}_1, \hat{P}_2|_{\bar{\mu}'} + \int_{\bar{\Omega}'} \|\alpha_1, \alpha_2\| \, d_{\bar{\mu}'} < 2\varepsilon_1 + 2\varepsilon_2.$$

Continue inductively building $\hat{P}_i$, α_i with

$$\bar{d}^G((\bar{T}, P, g_{\bar{\omega}}); (\bar{T}', \hat{P}_i, \alpha_{i\bar{T}'(\bar{\omega}')} \circ g'_{\bar{\omega}'} \circ \phi^{-1}(\alpha_{i\bar{\omega}'}^{-1}))) < 2\varepsilon_i,$$

and

$$|\hat{P}_i, \hat{P}_{i+1}|_{\bar{\mu}'} + \int_{\bar{\Omega}'} \|\alpha_i, \alpha_{i+1}\| \, d\bar{\mu}' < 2\varepsilon_i + 2\varepsilon_{i+1}.$$

Hence $\hat{P}_i \to \bar{P}$ and $\alpha_i \to \alpha$. ∎

THEOREM 3. *If $(\bar{T}, P, g_{\bar{\omega}})$ and $(\bar{T}', P', g'_{\bar{\omega}'})$ are two G-finitely determined G-processes, $h(\bar{T}, P) = h(\bar{T}', P') < \infty$, then there are $\bar{P}$, $\alpha \subset \mathcal{O}(T'_g)$ with*

$$(\bar{T}, P, g_{\bar{\omega}}) \equiv (\bar{T}', \bar{P}, \alpha_{\bar{T}'(\bar{\omega}')} \circ g'_{\bar{\omega}'} \circ \phi^{-1}(\alpha_{\bar{\omega}}^{-1}))$$

as G-processes and $\bigvee_{i=-\infty}^{\infty} T'^i(\bar{P}) = (T'_g)$.

Proof. Modify the proof of the Z-action isomorphism Theorem exactly as Theorem 2 modifies the proof of Sinai's Theorem for Z-actions (see [3] or [5] for proofs for Z-actions). ∎

This now completes our isomorphism theorem. A glaring question in our construction is whether there even exist G-finitely determined G-actions. The answer to this is yes; the construction, though, will appear elsewhere in a joint work with J. Feldman. Knowing this, though, we can show that our G-finitely determined actions are Bernoulli. The standard definition of a Bernoulli group action is one for which all its infinite discrete subgroup actions are isomorphic to the corresponding shift action on i.i.d. random variables of the proper entropy. We will now show that any G-finitely determined G-action satisfies this.

LEMMA 9. *If $(\bar{T}, g_{\bar{\omega}})$ is a G-finitely determined G-action, $h(\bar{T}) < \infty$, then for any $\bar{g} \in \bar{G}$, and $n \geqslant 1$, $T_{\bar{g}} \circ T_1^n$ is Bernoulli.*

Proof. Construct a $G' = Z \otimes^{\phi^n \circ \bar{g}^*} \bar{G}$ action $(\bar{g}^*(\bar{g}') = \bar{g}^{-1}\bar{g}'\bar{g})$ $\{\hat{T}_g\}_{g \in G}$ with $\hat{T}_1$ Bernoulli with the same entropy at T_1^n. Now define $\tilde{T}_1 = \hat{T}_{\phi^n(\bar{g})} \circ \hat{T}_1$, and $\tilde{T}_{\bar{g}} = \hat{T}_{\bar{g}}$, $\bar{g} \in \bar{G}$. Now $\tilde{T}_1 \circ \tilde{T}_{\bar{g}} = \tilde{T}_{\phi^n(\bar{g})} \circ \tilde{T}_1$, hence this is a free $Z \otimes^{\phi^n} \bar{G}$-action with the same entropy as $\{T_1^{kn} \circ T_{\bar{g}}\}_{(kn, \bar{g}) \in Z \otimes^{\phi^n} \bar{G}}$. As T_1^n is Bernoulli, by Theorems 1 and 2, we can imbed this $Z \otimes^{\phi^n} \bar{G}$ action as a factor of $\{\tilde{T}_g\}$. This makes $T_{\phi^{-n}(\bar{g})} \circ T_1^n$ a factor of $\hat{T}_1^n$, and hence Bernoulli. We are done as $\phi^{-n}(\bar{g})$ can be anything. ∎

COROLLARY 1. *Any infinite discrete subgroup action of a G-finitely determined G-action is isomorphic to the shift action on i.i.d. random variables of the proper entropy.*

Proof. Any discrete subgroup H of $Z \otimes^{\phi} \bar{G}$ is of the form

$$\{(n_0, g_0)^i \circ (0, h), h \in \bar{H}, i \in Z\},$$

where $\bar{H} \subset \bar{G}$ is a finite subgroup and $n_0 \neq 0$. By Lemma 9, $T_{(n_0, g_0)}$ is Bernoulli, and the result follows from Theorems 1 and 3. ∎

Some interesting questions remain open at this point. Does the isomorphism theorem hold for infinite entropy? This is equivalent to asking whether $g_{\bar{\omega}}$ can be made measurable with respect to a finite entropy factor of $\bar{T}$. This can be done when ϕ is an isometry, but in general the result is not clear. Can one extend this structure further and define a notion of very weak Bernoulli for such actions? This would be very useful to try to prove analogues of the results in [6] for more general extensions. Finally, can this kind of topological extension of the Isomorphism Theory be applied to a more general group action? In its most extreme, say a group G which has a discrete co-compact subgroup D to which the Isomorphism Theory applies. As an example, this structure *can* be used to give the isomorphism theorem for $\mathbb{R}$ actions.

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